Dynamical zero-temperature phase transitions and cosmic inflation/deflation

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For a rather general class of scenarios, sweeping through a zero-temperature phase transition by means of a time-dependent external parameter entails universal behavior: In the vicinity of the critical point, excitations behave as quantum fields in an expanding or contracting universe. The resulting effects such as the amplification or suppression of quantum fluctuations (due to horizon crossing, freezing, and squeezing) including the induced spectrum can be derived using the curved space-time analogy. The observed similarity entices the question of whether cosmic inflation itself might perhaps have been such a phase transition.

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In contrast to thermal phase transitions occurring when the strength of the thermal fluctuations equals a certain threshold (and so changes the character of the stable phase), zero-temperature phase transitions such as quantum phase transitions [1] denote the crossover of different ground states at a certain (critical) value of some external parameter, where quantum fluctuations play a dominant role, cf. Fig. 1. In both cases, there exists a vast amount of literature regarding the equilibrium properties in the vicinity of the phase transition, for example in view of universal behavior (e.g., scaling laws) near the critical point. However, since response times typically diverge in the vicinity of the critical point, sweeping through the phase transition with a finite velocity, for example, leads to a break-down of adiabaticity and thus might generate interesting dynamical (non-equilibrium) effects. For thermal phase transitions, a prominent example is the Kibble-Zurek mechanism, i.e., the generation of topological defects via rapid cooling (quench), which can be applied to the phase transitions in the early universe as well as in the laboratory [2].

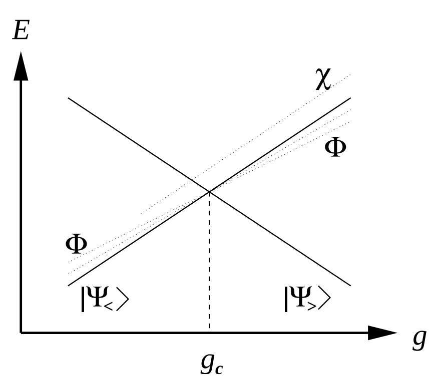

FIG. 1: Sketch of level structure near the critical point. The energy E of the levels is plotted as a function of some external parameter g. At the critical point $g = g_c$, the ground state changes from $|\Psi_<\rangle$ to $|\Psi_>\rangle$. Typical excitations above the ground state $|\Psi_<\rangle$ for $g < g_c$ are denoted by Φ and χ. Some of the excitations χ may remain stable after crossing the critical point $g = g_c$, whereas others Φ become unstable i.e., lie below $|\Psi_<\rangle$.

The following considerations are devoted to non-equilibrium effects in a rather general class of zero-temperature transitions, which also expose a remarkable analogy between cosmology and laboratory physics, see also [3]. Let us consider a quantum system (at zero temperature) described by the Hamiltonian $\hat{H}$ depending on some external parameter g. At a certain critical value of this parameter g_c, the system is supposed to undergo a phase transition, i.e., the ground state $|\Psi_<(g)\rangle$ of $\hat{H}(g)$ for $g < g_c$ is different from the ground state $|\Psi_>(g)\rangle$ of $\hat{H}(g)$ for $g > g_c$. For example, $|\Psi_<(g)\rangle$ and $|\Psi_>(g)\rangle$ could have different global/topological properties (such as magnetization) in the thermodynamic limit.

Before the phase transition $g < g_c$, the phase space can be explored in terms of quasi-particle excitations above the ground state $|\Psi_<(g)\rangle$. If we make the assumption that the Hamiltonian $\hat{H}$ is analytic in the external parameter g, we may follow the behavior of these excitations through the critical point g_c and extrapolate them to the region $g > g_c$. For $g > g_c$, the state $|\Psi_<(g)\rangle$ is no longer the ground state and hence some of the excitations must become unstable after the critical point $g > g_c$, since now $|\Psi_>(g)\rangle$ has a lower energy than $|\Psi_<(g)\rangle$, cf. Fig. 1. In the following, we shall focus on these quasi-particle excitation modes, whose energy is positive for $g < g_c$ and becomes negative after crossing the critical point $g > g_c$. In order to describe these excitations quantitatively, a few additional assumptions/approximations are necessary:

Linearity: We assume that the crossover from stability to instability of these modes can be described using linear stability analysis, i.e., that the associated quantum fluctuations are small enough. Consequently, the onset of instability occurs if the dispersion relation $\omega^2(k)$ dives below the k-axis.

Vanishing gap: Since we are mainly interested in low-energy and long-wavelength excitations (which will turn out to yield universal behavior largely independent of the microscopic structure), we shall assume that the modes Φ are gap-less such as Goldstone modes, i.e., $\omega^2(k = 0) = 0$ for all g. The analogy to quantum fields in an expanding/contracting universe applies to the general case with a gap as well, but the main points of interest (analogue

of cosmic horizon) can be studied for gap-less modes.
Analyticity: The dispersion relation $\omega^2(k,g)$ is supposed to be an analytic function of k. Hence a Taylor expansion starts with the term [with $c^2(g<g_c)>0$]

$$\omega^2(k,g) = k^2c^2(g) + \mathcal{O}(k^3)\,. \tag{1}$$

Independence: As the final ingredient, we assume that the unstable modes are independent of each other such that it suffices to consider one scalar mode Φ.

Based on these assumptions, we may construct the low-energy effective theory describing the mode Φ in the vicinity of the critical point g_c according to the dispersion relation (1). As demonstrated in Ref. [4], given the above conditions, the most general low-energy effective action can be cast into the form

$$\mathcal{L}_{\rm eff} = \frac{1}{2}\sqrt{-g_{\rm eff}}\,g^{\mu\nu}_{\rm eff}(\partial_\mu\Phi)(\partial_\nu\Phi)\,, \tag{2}$$

where $\sqrt{-g_{\rm eff}}\,g^{\mu\nu}_{\rm eff}$ denotes a matrix depending on the system under consideration as well as the external parameter and a sum convention over $\mu,\nu=0\dots3$ is employed.

Hence the excitations Φ behave as minimally coupled and mass-less scalar quantum fields in curved space-times whose geometry is determined by the effective metric $g^{\mu\nu}_{\rm eff}$; which is basically the underlying idea of the analogue gravity concept, see, e.g., [5, 6, 7]. If we assume the quantum system under consideration to be effectively homogeneous and isotropic at large wavelengths λ, the above action simplifies to

$$\mathcal{L}_{\rm eff} = \frac{1}{2}\left(\frac{1}{\alpha}\,\dot\Phi^2 - \beta\,(\boldsymbol{\nabla}\Phi)^2\right)\,. \tag{3}$$

The external parameters α and β may depend on g and hence on time and must be non-negative before the phase transition. The convenience of the choice of $1/\alpha$ in the first term becomes apparent after constructing the associated effective Hamiltonian

$$\mathcal{H}_{\rm eff} = \frac{1}{2}\left(\alpha\,\Pi^2 + \beta\,(\boldsymbol{\nabla}\Phi)^2\right)\,. \tag{4}$$

In the homogeneous and isotropic case, the effective metric reads (for $g<g_c$)

$$ds^2_{\rm eff} = \sqrt{\alpha\beta^3}\,dt^2 - \sqrt{\beta/\alpha}\,d\boldsymbol{r}^2\,. \tag{5}$$

Since the mode Φ becomes unstable for $g>g_c$, at least one of the two parameters has to change its sign at the critical point. There are basically three possibilities:
A: $\alpha\downarrow 0$ while β remains finite. (Note that $1/\alpha\downarrow 0$ is not possible since the propagation speed would diverge.) According to Eq. (5), the effective metric corresponds to an expanding universe in this case.
B: $\beta\downarrow 0$ while α remains finite (similarly, $1/\beta\downarrow 0$ is not possible), which corresponds to a contracting universe.
C: Both α and β become singular.

Having established the analogy to cosmology, we may now apply the tools and concepts known from curved space-times [8]. The space-time described by the above metric contains a (particle) horizon if the maximum distance (measured in co-moving coordinates) which can be traveled starting at the time $t_{\rm in}$

$$\Delta r = \int\limits_{t_{\rm in}}^{t_{\rm out}} dt\,c(t) = \int\limits_{t_{\rm in}}^{t_{\rm out}} dt\,\sqrt{\alpha(t)\beta(t)}\,, \tag{6}$$

is finite, i.e., points beyond the horizon size Δr can never be reached. Obviously, since $c=\sqrt{\alpha(t)\beta(t)}$ is bounded from above, a horizon always exists if the critical point is reached in a finite laboratory time $t_{\rm out}$ (which might still correspond to an infinite proper time in cosmology), but also if α and/or β decrease fast enough for $t_{\rm out}\uparrow\infty$, for example $\alpha\propto 1/t^n$ with $n>2$, cf. [9].

As one can infer from the above equation, the horizon size (in terms of the laboratory coordinates $t,\boldsymbol{r}$) always decreases as $t_{\rm in}$ increases. Hence, all Φ-modes with a given (effective) wavelength λ cross the horizon at some time (when λ exceeds the horizon size Δr). After this horizon crossing, the modes cannot oscillate anymore (freezing) due to loss of causality across the horizon and their quantum state gets squeezed. This mechanism is completely analogous to the amplification of quantum vacuum fluctuations of the inflaton field in our present standard model of cosmology – which are supposed to be the seeds for structure formation.

In order cast this analogy into a more quantitative form, let us consider the equations of motion for Φ

$$\left(\frac{\partial}{\partial t}\,\frac{1}{\alpha(t)}\,\frac{\partial}{\partial t} - \beta(t)\,\boldsymbol{\nabla}^2\right)\Phi = 0\,. \tag{7}$$

In the homogeneous and isotropic case $\boldsymbol{\nabla}\alpha=\boldsymbol{\nabla}\beta=0$, there is a duality between the field Φ and its canonical momentum $\Pi=\dot\Phi/\alpha$ since Π obeys the same the equation of motion as Φ with α and β interchanged

$$\left(\frac{\partial}{\partial t}\,\frac{1}{\beta(t)}\,\frac{\partial}{\partial t} - \alpha(t)\,\boldsymbol{\nabla}^2\right)\Pi = 0\,. \tag{8}$$

This duality is analogous to that between the electric and magnetic field in the absence of macroscopic sources. However, even though the equations of motion in cases **A** and **B** are related by this duality, the behavior of the quantum fluctuations (e.g., their spectrum) is different.

Assuming general power-law time-dependence near the critical point $\alpha\propto t^a$ and $\beta\propto t^b$, we may remember the coordinate invariance of the effective geometry and introduce another time coordinate τ proportional to $t^{2/(2+a+b)}$, for which the wave equation simplifies to

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{2\nu-1}{\tau}\,\frac{\partial}{\partial\tau} - \boldsymbol{\nabla}^2\right)\Phi = 0\,, \tag{9}$$

provided that ν is given by

$$\nu = \frac{1+a}{2+a+b}\,. \tag{10}$$

Hence the solutions for the spatial Fourier modes can be expressed in terms of Hankel functions [10]

$$\Phi_k^\pm(\tau) = \tau^\nu\, H_\nu^\pm(k\tau)\,. \tag{11}$$

Sufficiently far away from the critical point $|\tau| \uparrow \infty$, the modes oscillate $\Phi_k^\pm(\tau) \sim \tau^\nu\, e^{\pm ik\tau}/\sqrt{k\tau}$ and the solutions $\Phi_k^\pm$ are basically the functions in the expansion into creation and annihilation operators corresponding to the adiabatic vacuum state, cf. [8]. When approaching the phase transition, however, adiabaticity breaks down and the modes cross the horizon and freeze [10]

$$\Phi_k^\pm(|\tau| \downarrow 0) \sim k^{-\nu}\,. \tag{12}$$

I.e., the spectrum of the two-point correlation function $\langle\hat\Phi(\boldsymbol{r})\hat\Phi(\boldsymbol{r'})\rangle$ of the frozen quantum fluctuations is determined by the parameter $\nu > 0$ in Eq. (10).

Let us examine a few examples: The trivial case $a = b = 0$ of course reproduces the undisturbed spectrum $\nu = 1/2$. Neglecting the back-reaction of the quantum fluctuations onto the the dynamics of the external parameter g, its time-evolution $g(t)$ should not experience anything special at the critical point $g = g_c$ and hence the most natural choice for its dynamics is a constant velocity $g - g_c \propto t$. If we assume the effective Hamiltonian to be an analytic function of g, this implies $a = 1$ and/or $b = 1$. (Otherwise a and b are the characteristic exponents $|g - g_c|^a$ and $|g - g_c|^b$ occurring in $\hat H$.) In case **A** (expanding universe) with $a = 1$ and $b = 0$, we obtain $\nu = 2/3$, i.e., quantum fluctuations with large wavelengths are amplified. case **B** (contracting universe) with $a = 0$ and $b = 1$ yields $\nu = 1/3$, i.e., quantum fluctuations with large wavelengths are suppressed. Finally, $a = b = 1$ (case **C**) leads to an undistorted spectrum $\nu = 1/2$, which is not surprising once one realizes that the effective metric in Eq. (5) is exactly flat in terms of the new time-coordinate $ds^2_{\rm eff} = d\tau^2 - d\boldsymbol{r}^2$.

Now we are in the position to apply the above method to some concrete physical systems: As a first example, we consider atomic Bose-Einstein condensates. In the dilute-gas limit, the quantum phase and density fluctuations are small and can be treated as linear perturbations. For wavelengths far above the healing length, the effective action of the phase fluctuations Φ reads ($\hbar = 1$)

$$\mathcal{L}_{\rm eff} = \frac{1}{2}\left(\frac{1}{g}\,\dot\Phi^2 - \frac{\varrho_0}{m}\,(\boldsymbol{\nabla}\Phi)^2\right)\,, \tag{13}$$

where ϱ_0 is the background density of the condensate, m the mass of the atoms, and g the time-dependent coupling strength representing the inter-particle repulsion $g > 0$ or attraction $g < 0$. Obviously, a homogeneous condensate becomes unstable for attractive interactions (i.e., $g_c = 0$) and this scenario $g \downarrow 0$ corresponds to case **A**, i.e., an expanding universe. Hence sweeping through the phase transition at $g_c = 0$ by means of a time-dependent external magnetic field (Feshbach resonance) with a finite velocity generates a $k^{-4/3}$-spectrum of the two-point phase-phase correlation function. However, it might be difficult to measure the phase after the phase transition (collapse of condensate due to molecule formation), whereas the frozen density fluctuations with large wavelengths should easily be measurable. The density fluctuations are just the canonically conjugated momentum field Π and can be calculated analogously using the duality in Eqs. (7) and (8), but with an additional factor of k in Eq. (11). Hence the spectrum of the density-density correlation function behaves as $k^{+4/3}$. Note that this behavior is consistent with the amplification/suppression of quantum fluctuations by squeezing which maintains the minimal Heisenberg uncertainty of the ground state, i.e., $\Delta q_k \Delta p_k = \hbar/2$.

As a second example, let us study a simple 1+1 dimensional model of the electromagnetic field coupled to a linear medium via the magnetic component

$$\mathcal{L}_{\rm eff} = \frac{1}{2}\left(E^2 - B^2 + \dot\Psi^2 - \Omega^2\Psi^2 + gB\Psi\right)\,, \tag{14}$$

with the electromagnetic field being governed by the potential A via $E = \partial_t A$ and $B = \partial_x A$. The field Ψ describes the (linearized and localized) dynamics of the medium with the plasma frequency Ω and g denotes the coupling (magnetic dipole moment). Averaging over the degrees of freedom Ψ of the medium, the low-energy effective theory for the macroscopic electromagnetic field yields the permeability $1/\mu = 1 - g^2/\Omega^2$ (which corresponds to inserting the adiabatic solution $\Psi \approx gB/\Omega^2$ back into the action). Hence there is a phase transition at the critical value of the coupling $g_c = \Omega$ after which the medium develops a spontaneous magnetization and the linearized description above breaks down. Identifying $A = \Phi$, this scenario corresponds to case **B**. Hence the frozen two-point spectra behave as $k^{-2/3}$ for A, and thus $k^{4/3}$ for B (and Ψ), and finally $k^{2/3}$ for $E = \Pi$ (again respecting $\Delta q_k \Delta p_k = \hbar/2$) if we sweep through the critical point with a finite velocity. Note that, for the source-free macroscopic electromagnetic field, one can also introduce the dual potential Λ via $H = B/\mu = \partial_t\Lambda$ and $D = E = \partial_x\Lambda$, which explicitly incorporates the duality in Eqs. (7) and (8), i.e., case **B** $\to$ case **A**, and consequently has a $k^{-4/3}$-spectrum.

Finally, as a third example, we consider a very simple version of the Heisenberg model

$$H = -g(t)\sum_{<ij>}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\,, \tag{15}$$

with $\boldsymbol{\sigma}_i$ being the Pauli spin-1/2 operators for each lattice site i and $< ij >$ denoting the sum over nearest and

next-to-nearest neighbors, for example. The ferromagnetic state $|\Psi_<\rangle = |\dots\uparrow\uparrow\uparrow\uparrow\dots\rangle$ is the ground state for $g > 0$ and breaks the $O(3)$ invariance of the Hamiltonian; thus the spin-waves (magnons) are gap-less (Goldstone theorem). Obviously $g = 0$ is the critical point here, after which the energy of the magnons becomes negative. In view of the global factor $g(t)$, this scenario is an example for case **C** with $\alpha = \beta = g$, i.e., $a = b$. The observation that the spectrum of the fluctuations is not disturbed by the dynamics of $g(t)$ can also be explained by the fact that the ferromagnetic state $|\dots\uparrow\uparrow\uparrow\uparrow\dots\rangle$ is an exact eigenstate of the Hamiltonian, i.e., even the decay from $|\Psi_<\rangle$ to $|\Psi_>\rangle$ is impossible with the exact Hamiltonian in Eq. (15) and requires some disturbances.

In summary, the analogy between the excitations Φ and quantum fields in an expanding/contracting universe allows us to apply universal geometrical concepts such as horizons to a rather general class of quantum systems approaching the critical point. Near the phase transition, adiabaticity breaks down (since the energy gap vanishes) and the system does not stay in its (instantaneous/adiabatic) ground state in general. The spectrum of the two-point correlation function of the quantum fluctuations frozen out at the phase transition (which can be the seeds for pattern formation etc.) can be derived quite independent of the microscopic details of the considered system and is basically determined by the characteristic exponents a and b (universal behavior). The strength of the frozen fluctuations, however, and their dynamics after crossing the critical point (nonlinear instabilities etc.) depend on the explicit microscopic structure (e.g., the diluteness parameter in Bose-Einstein condensates).

As an outlook, one might compare phase transitions (such as the examples considered above) to "real" cosmic inflation which is part of the present standard model of cosmology. Interestingly, phase transitions reproduce many features of inflation: The decay of $|\Psi_<\rangle$ down to $|\Psi_>\rangle$ at $g > g_c$ (breakdown of adiabaticity) releases energy (analogous to reheating). Phase transitions display universal behavior (no fine-tuning) in the sense that initial small-scale deviations from $|\Psi_<\rangle$ are not important after the transition $g > g_c$. Similarly, the large-scale homogeneity may be explained naturally. However, even though quantum fluctuations generate small inhomogeneities in both scenarios (phase transitions and "real" cosmic inflation), none of the examples considered above reproduces the correct scale-invariant $1/k^3$-spectrum. However, that is not surprising as the examples considered above break many symmetries we observe in the real universe, e.g., they possess a preferred frame and do not respect the principle of equivalence etc. If we *demand* that the effective action (at least at low energies) does not single out a locally preferred frame (e.g., the two-point function $\langle\hat{\Phi}(\underline{x})\hat{\Phi}(\underline{x}')\rangle$ depends on the Ricci scalar etc.) there are only two possibilities: firstly, the effectively flat space-time with $a = b$ leading to an undisturbed $1/k$-spectrum, and, secondly, the remaining nontrivial combination $a + 3b = -4$ exactly corresponds to the de Sitter metric and thus reproduces the scale-invariant $1/k^3$-spectrum, cf. Eq. (10) and [9]. The second version allows for more additional symmetries: We may assume a constant velocity of propagation for the Φ-mode

$$\mathcal{A} = \int dt\, d^3r\, \frac{1}{2}\, \frac{\dot{\Phi}^2 - (\boldsymbol{\nabla}\Phi)^2}{t^2}\,, \tag{16}$$

and the resulting action turns out to be scale-invariant: $\mathcal{A}[t \to \lambda t, \boldsymbol{r} \to \lambda\boldsymbol{r}] = \mathcal{A}[t, \boldsymbol{r}]$. (There are also other motivations for the above form of the action such as the principle of equivalence, which shall not be discussed here.) One would expect the dynamics of this action, which has been motivated by *demanding* the above symmetries, to be generated by the back-reaction, which has been omitted so far and which respects these symmetries. These interesting findings entice the question/speculation of whether cosmic inflation itself might perhaps have been such a phase transition.

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